STANFORD RESEARCH INSTITUTE
MENLO PARK, CALIFORNIA

RPN14

September 1964

Technical Report 204

ELECTROMAGNETIC WAKE OF A CHARGED PARTICLE PULSE IN A PLASMA

Prepared for:

OFFICE OF NAVAL RESEARCH
WASHINGTON 25, D.C.

CONTRACT Nonr 2728(00)

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ABSTRACT

Employing the hydrodynamic approach, the electromagnetic wake of a charged particle pulse passing through a plasma is investigated. The perturbation in the magnetic field amplitude $B_\theta^{(1)}$ which is of interest in the study of "hose" instability of the beam is evaluated. The effect of charge density profile and different velocity regimes of the beam pulse on the behavior of $B_\theta^{(1)}$ is considered. Finally, numerical results relevant to a typical example are presented.

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I INTRODUCTION

GENERAL

In problems concerned with the hose instability[1] of a beam of charged particles penetrating a plasma (conducting) medium, it is of interest to know the decay distances of the magnetic field amplitudes* due to a finite beam pulse. The behavior of the electromagnetic wake of such a pulse will be analyzed here employing the hydrodynamic approach.

Some calculations have been made in the past by S. Chandrasekhar regarding the electromagnetic wake following a pulse of charged particles in a plasma.[2] We will employ here the M.K.S. system of units. The present report may be considered as an extension and in some respects a generalization of the earlier work of Chandrasekhar.

COMMENTS ON THE CHOICE OF MODELS

Before we proceed with the analysis of the electromagnetic wake problem, we note that the models studied here are different from the one employed by Chandrasekhar.

CHANDRASEKHAR'S MODEL

Assume that, in the unperturbed state, the beam pulse is describable as a current $\mathbf{J}_{oo}(\mathbf{r})$ penetrating the plasma (See Fig. 1); then,

$$\mathbf{J}_{oo}(\mathbf{r}) = q_o(\mathbf{r})\ \mathbf{v}\ \ldots \tag{1}$$

where $q_o(\mathbf{r})$ and $\mathbf{v}$ are the charge density and velocity of the beam (assumed to be uniform), respectively; then, in the rest frame of the pulse, one may write

* The nature of these will become clearer in the sequel.

[1] To understand the nature of this instability, see, for instance, M. N. Rosenbluth, *J. Phys. Fluids*, 3, 392 (1960), and S. Weinberg: "The Hose Instability Dispersion Relation" (to be published).

[2] S. Chandrasekhar: Unpublished work (1961)

$$q_0(r) = q_t(\rho,\theta)q_{\|}(z) \quad \ldots \tag{2}$$

where $\rho, \theta,$ and z define the cylindrical coordinate system.

He assumed that

$$q_{\|}(z) = \begin{cases} \text{constant}, & -L/2 < z < L/2 \\ 0, \text{ otherwise} & \end{cases} \tag{3}$$

and

$$q_t(\rho,\theta) = \begin{cases} q_t^{(0)} + q_t^{(1)}\rho e^{i\theta}, & \rho \leq R \\ 0, \ \rho > R & \end{cases} \quad \ldots \tag{4}$$

and obtained *explicit* expressions for the magnetic field amplitudes in the wake for $|v| = c$, where c is the speed of light.

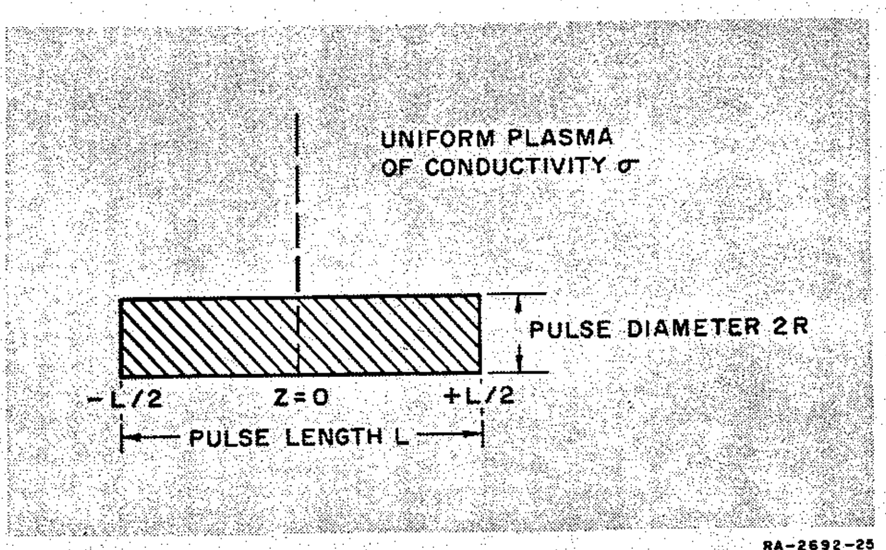

FIG. 1 SCHEMATIC OF THE PROBLEM TREATED

PRESENT CHOICE OF MODELS

The following remarks will serve to explain the reasons for the present choice of models in the sequel. We start with Eq. (1). We now assume that the beam is rigidly displaced by an amount $\mathbf{d}$ (say) in the lateral direction. Then, a beam particle initially located at a radius ρ will in a time interval Δt occupy a different position according to the relation

$$[\rho - \mathbf{d}(z,t,\theta), z] \rightarrow [\rho - \mathbf{d}(z + \mathbf{v}\Delta t, t + \Delta t, \theta), z + \mathbf{v}\Delta t] \ \ldots \qquad (5a)$$

We also know that

$$\mathbf{v}(z,t) \rightarrow \mathbf{v} + \left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial z}\right)\mathbf{d} \ \ldots \qquad (5b)$$

and,

$$q_o(\rho, z, t) \rightarrow q_o(\rho - \mathbf{d}, z, t) \ \ldots \qquad (5c)$$

If we now assume that

$$|\mathbf{d} \cdot \nabla q_0| << q_0$$

$$\left|\frac{\partial \mathbf{d}}{\partial t}\right| << |\mathbf{v}|$$

$$\left|\frac{\partial \mathbf{d}}{\partial z}\right| << 1 \ ,$$

then, to first order we obtain the following, employing the relation for the conservation of charge. If $q_1(\mathbf{r})$ is the perturbation in the charge density of the beam due to the *small* rigid displacement $\mathbf{d}$, then

$$q_1(\rho, z, t) = q_1(\mathbf{r}) \approx - [\mathbf{d} \cdot \nabla q_0(\mathbf{r})] = - [\mathbf{d} \cdot \nabla_t q_0(\rho)] \qquad (6)^*$$

Now, if the rigid displacement $\mathbf{d}$ in the lateral direction is Fourier analyzed as

$$\mathbf{d} = \sum_{n=0}^{\infty} \mathbf{d}^{(n)} e^{in\theta} \ \ldots \qquad (7a)$$

* Since q_0 is a function of ρ only and not a function of z.

then,

$$q_1(\mathbf{r}) \approx - \left[\sum_{n=0}^{\infty} \mathbf{d}^{(n)} e^{in\theta} \right] \cdot \nabla q_0(\mathbf{r}) \quad \ldots \qquad (7b)$$

or, one may write

$$q_1(\mathbf{r}) \approx \sum_{n=0}^{\infty} q^{(n)} e^{in\theta} \quad \ldots \qquad (7c)^*$$

We will remark that, in problems dealing with hose instability, one uses only the second term of the series in (7c); that is, the term where $n = 1$ is the one of interest to us.

The charge distribution $q(\mathbf{r})$ in a frame moving with the beam at velocity $\mathbf{v}$ may be expressed as

$$q(\mathbf{r}) = q_t(\rho,\theta) \cdot q_{\|}(z) \quad \ldots \qquad (8)$$

Although the formulae obtained in this report, following the procedure of Chandrasekhar, are valid for an arbitrary distribution of charge density along the z-direction, it will be assumed here that

$$q_{\|}(z) = \begin{cases} \text{constant, (say unity)}, & -L/2 \le z \le L/2 \\ 0, & \text{otherwise.} \end{cases} \qquad (9)$$

That is, the pulse is of length L. The choice of $q_t(\rho,\theta)$ will be considered next.

Since, in practice, one usually obtains a beam whose charge density varies with ρ, we will choose two types of models.

MODEL I

This we call the 'ρ^2' beam and it appears to be close to reality. Here, the unperturbed charge density $q_t(\rho,\theta)$ is given by

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* Compare S. Weinberg, *Op. cit.*

$$q_t(\rho,\theta) \sim \begin{cases} (1 - \rho^2/R^2), \text{ for } \rho \le R \\ 0, \text{ otherwise} \end{cases} \tag{10}$$

where R is the radius of the beam (that is, we assume a well-defined beam edge).

In the perturbed state (due to the rigid displacement), one may write

$$q_t(\rho,\theta) = q_t^{(0)} + q_t^{(1)}$$

where $q_t^{(0)}$ and $q_t^{(1)}$ are the unperturbed and perturbation charge distributions of the beam

$$q_t^{(0)} = \begin{cases} q^{(0)}(1 - \rho^2/R^2), \text{ for } \rho \le R \\ 0, \text{ otherwise} \end{cases} \tag{11a}$$

$$q_t^{(1)} = \begin{cases} q^{(1)} \cdot \rho \cdot e^{i\theta}, \text{ for } \rho \le R \\ 0, \text{ otherwise} \end{cases} \tag{11b}$$

Note that (11b) assumes $|d| << R$.

MODEL II

To find if the character of the electromagnetic wake is sensitive to beam charge distribution, and for the purpose of completeness, we will obtain some results employing Model II.

Let us call this the "triangular" beam. Here

$$q_t(\rho,\theta) \sim \begin{cases} (1 - \rho/R), \text{ for } \rho \le R \\ 0, \text{ otherwise.} \end{cases} \tag{12}$$

Employing again the previous notation

$$q_t(\rho,\theta) = q_t^{(0)} + q_t^{(1)}$$

we have

$$q_t^{(0)} = \begin{cases} q^{(0)}(1 - \rho/R), & \text{for } \rho \leq R \\ 0, & \text{otherwise} \end{cases} \tag{13a}$$

$$q_t^{(1)} = \begin{cases} q^{(1)}e^{i\bar{\theta}}, & \text{for } \rho \leq R \\ 0, & \text{otherwise.} \end{cases} \tag{13b}$$

Again, (13) implies $|d| << R$.

In addition to the above choice of models, we will treat below three velocity regimes of the beam. That is, employing Model I, we study the following cases:

Case 1: $|v| = c$
Case 2: $|v| \approx c$
Case 3: $|v| << c$

and employing Model II, we will obtain results for the case where $|v| = c$, and we will comment on the other cases.

II DESCRIPTION OF THE ELECTROMAGNETIC WAKE

One starts with Maxwell's equations

$$\left.\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \;;\quad \nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0;\quad \mathbf{B} = \mu_0 H;\quad \mathbf{D} = \epsilon_0 \mathbf{E} \\ \nabla \cdot \mathbf{D} &= q\;;\quad \mathbf{j} = \sigma \mathbf{E} + q\mathbf{v} \end{aligned}\right\} \tag{14}$$

where q is the beam charge density, and we assume charge neutrality of the plasma. Where σ is the conductivity of the plasma, which we assume to be a scalar quantity in the present treatment. The rest of the symbols denote the usual field quantities, etc. The current $\mathbf{j}$ consists of two components: one corresponding to the current induced in the plasma and the other due to the motion of charges of the beam. Also,

$$c^2 = \frac{1}{\mu_0 \epsilon_0} \quad .$$

It follows from Eq. (13) that

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{H}), \text{ or}$$

$$-c^2(\nabla \times \nabla \times \mathbf{E}) = \frac{1}{\epsilon_0}\left(\sigma \frac{\partial \mathbf{E}}{\partial t} + \mathbf{v}\frac{\partial q}{\partial t}\right) + \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad . \tag{15}$$

In the sequel, $\mathbf{v}$ will be treated as an approximately constant quantity. Employing the Fourier transforms technique, one writes

$$\left.\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \int_{-\infty}^{\infty} d\mathbf{k}\,\vec{\mathcal{E}}(\mathbf{k}) \exp i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t) \\ \mathbf{B}(\mathbf{r},t) &= \int_{-\infty}^{\infty} d\mathbf{k}\,\vec{\mathcal{B}}(\mathbf{k}) \exp i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t) \\ q(\mathbf{r}) &= \int_{-\infty}^{\infty} d\mathbf{k}\,q(\mathbf{k}) \exp i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t) \end{aligned}\right\} \qquad (16)^{*}$$

Substitution of relations (16) in Eq. (15) yields

$$c^2[\mathbf{k}(\mathbf{k}\cdot\vec{\mathcal{E}}) - k^2\vec{\mathcal{E}}] = \left[-\frac{i}{\epsilon_0}(\mathbf{k}\cdot\mathbf{v})\sigma + (\mathbf{k}\cdot\mathbf{v})^2\right]\vec{\mathcal{E}} - \frac{i}{\epsilon_0}(\mathbf{k}\cdot\mathbf{v})q(\mathbf{k})\cdot\mathbf{v} \qquad (17)$$

Multiplying Eq. (17) vectorially by $\mathbf{k}$, one obtains

$$\mathbf{k}\times\vec{\mathcal{E}} = \frac{-\dfrac{i}{\epsilon_0}(\mathbf{k}\cdot\mathbf{v})(\mathbf{k}\times\mathbf{v})q(\mathbf{k})}{(\mathbf{k}\cdot\mathbf{v})^2 + \dfrac{i}{\epsilon_0}(\mathbf{k}\cdot\mathbf{v})\sigma - c^2k^2} \qquad (18)$$

Also, since

$$\vec{\mathcal{B}}(\mathbf{k}) = \frac{\mathbf{k}\times\vec{\mathcal{E}}}{(\mathbf{k}\cdot\mathbf{v})}$$

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* The following comments regarding the relations (16) are in order.

Usually, one writes

$$\mathbf{F}(\mathbf{r},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mathbf{k}\,d\omega\,\mathbf{F}(\mathbf{k},\omega) \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$$

where the frequency ω and wave number $\mathbf{k}$ are measured in the laboratory frame.

In our present treatment we are essentially considering a static problem and effects due to such things as Cerenkov radiation are ignored. If one looks at the Lorentz transformation $\omega' = (\omega - \mathbf{k}\cdot\mathbf{v})\gamma$. ω' measured in the rest frame of the pulse, $\gamma = (1-\beta^2)^{-1/2}$, and $\beta = v/c$ and notes that, when $\omega' = 0$, one obtains $\omega = \mathbf{k}\cdot\mathbf{v}$. That is, we have essentially, $\mathbf{F}(\mathbf{k},\omega) = \mathbf{F}(\mathbf{k})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$ and the representation employed in (16) is valid in our case when $|v| \to c$. If, however, ω' were not zero for some reason, one has to consider the effects due to Doppler phenomena, etc., by employing the conventional representation for $\mathbf{F}(\mathbf{k},\omega)$ indicated above.

$$\vec{\mathbb{B}}(\mathbf{k}) = \frac{\frac{i}{\epsilon_0}(\mathbf{k} \times \mathbf{v})q(\mathbf{k})}{\left[k^2c^2 - (\mathbf{k}\cdot\mathbf{v})^2 - \frac{i}{\epsilon_0}(\mathbf{k}\cdot\mathbf{v})\sigma\right]} \tag{19}$$

Equation (19) was obtained earlier by Chandrasekhar.

We will assume that the beam velocity is *only* z-directed; then, $\mathbf{v} = \mathbf{i}_z v$. Under that condition, it is obvious that

$$\vec{\mathbb{B}}(\mathbf{k}) = \frac{\frac{i}{\epsilon_0}(\mathbf{k} \times \mathbf{i}_z)q(\mathbf{k})v}{c^2\left[(k_x^2 + k_y^2) + k_\parallel^2\left(1 - \frac{v^2}{c^2}\right) - \frac{i}{\epsilon_0}k_\parallel\frac{\sigma}{c}\frac{v}{c}\right]} \tag{20}$$

where

$$k_\parallel^2 + k_y^2 + k_x^2 = k^2$$

If we write

$$k_x = k_t \cos\psi, \quad k_y = k_t \sin\psi$$

$$x = \rho\cos\theta, \quad y = \rho\sin\theta$$

Equation (20) is expressible as

$$\vec{\mathbb{B}}(k_t, k_\parallel, \psi) = \frac{\left(-i\mu_0 c\cdot k_t\cdot\frac{v}{c}\right)\cdot q(k_\parallel, k_t, \psi)}{(k_t^2 - iAk_\parallel + k_\parallel^2 B)}\cos(\psi - \theta) \tag{21}$$

where

$$B = \left(1 - \frac{v^2}{c^2}\right), \quad A = \frac{1}{\epsilon_0}\frac{\sigma}{c}\frac{v}{c}, \quad \text{and}$$

$$q(k_\parallel, k_t, \psi) = q_t(k_t, \psi)\cdot q_\parallel(k_\parallel)$$

We now employ the procedure used by Chandrasekhar, and write

$$q_t(k_t,\psi) = \frac{1}{4\pi^2}\int_0^\infty \rho d\rho \int_0^{2\pi} d\theta q_t(\rho,\theta)\left[(\exp - ik_t\rho \cos(\psi - \theta)\right]$$

$$q_\parallel(k_\parallel) = \frac{1}{2\pi}\int_{-\infty}^{\infty} q_\parallel(z)(\exp ik_\parallel z) \cdot dz^*,$$

and

$$q_t(\rho,\theta) = \sum_n q_t^{(n)}(\rho)e^{in\theta} \tag{22}$$

Let us now consider the field associated with each n separately. That is,

$$q_t^{(n)}(k_t,\psi) = \frac{1}{4\pi^2}\int_0^\infty \rho d\rho q_t^{(n)}(\rho)\int_0^{2\pi} d\theta \exp\left[- ik_t\rho \cos(\psi - \theta) + in\theta\right]$$

$$= \frac{1}{2\pi}q_t^{(n)}(k_t) \exp\left[in(\psi + 3\pi/2).\right] \tag{23}$$

where

$$q_t^{(n)}(k_t) = \int_0^\infty q_t^{(n)}(\rho)J_n(k_t\rho) \cdot \rho \cdot d\rho \tag{24}$$

Hence,

$$\vec{B}_\theta^{(n)}(k_t,k_\parallel,\psi) = \frac{-i\mu_0 ck_t \cdot \frac{v}{c}}{2\pi(k_t^2 - iAk_\parallel + k_\parallel^2 B)} q_t^{(n)}(k_t)q_\parallel(k_\parallel) \exp in(\psi + 3\pi/2) \tag{25}$$

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* Here z is written in place of $(z - vt)$. Hence, the description is in a frame at rest with respect to the pulse.

It is clear that

$$B_\theta^{(n)}(\rho,\theta,z) = \int_{-\infty}^{\infty} d\mathbf{k}\mathbb{B}^{(n)}(\mathbf{k}) \exp i\mathbf{k}\cdot\mathbf{r}$$

$$= \int_{-\infty}^{\infty} dk_\| e^{ik_\| z} \int_0^\infty k_t dk_t \int_0^{2\pi} d\psi \mathbb{B}_\theta^{(n)}(k_t,k_\|,\psi) \exp\left[ik_t\rho \cos (\psi-\theta)\right] \cdot \tag{26}$$

Assuming that changing the order of integration is permissible, we write

$$B_\theta^{(n)}(\rho,\theta\ z) = \frac{-i\mu_0 c}{2\pi} (v/c) e^{in(3\pi/2)} \int_0^\infty dk_t k_t^2 q_t^{(n)}(k_t) \int_{-\infty}^{\infty} \frac{dk_\| q_\|(k_\|) e^{ik_\| z}}{(k_t^2 - iAk_\| + k_\|^2 B)}$$

$$\times \int_0^{2\pi} d\psi \cos (\psi-\theta) \exp\left[ik_t\rho \cos (\psi-\theta) + in\psi\right] \cdots \tag{27}$$

Since

$$\int_0^{2\pi} \cos (\psi-\theta)\ e^{iz\cos(\psi-\theta)+in\psi} = -i\frac{d}{dz}\int_0^{2\pi} e^{iz\cos(\psi'-\theta)+in\psi'} d\psi'$$

$$= -i\frac{d}{dz}(2\pi)\cdot J_n(z) e^{in\theta}$$

$$2\ J_n'(z) = J_{n-1}(z) + J_{n+1}(z)$$

and

$$J_{-n}(z) = (-1)^n J_n(z) \cdot$$

We observe that

$$\int_0^{2\pi} d\psi \cos(\psi-\theta)\ e^{ik_t\rho\cos(\psi-\theta)+in\psi} = \pi\left[i^{-(n+1)}J_{-(n+1)}(k_t\rho) + i^{-n+1}J_{n+1}(k_t\rho)\right] e^{in\theta} \cdot \tag{28}$$

Hence we obtain—in agreement with Chandrasekhar—the following equation, which is a little more general.

$$B_\theta^{(n)}(\rho,z) = \frac{-i\mu_0 c}{2}\,\frac{v}{c}\cdot e^{in(3\pi/2)}\int_0^\infty dk_t k_t^2 q_t^{(n)}(k_t)\left[i^{-(n+1)}J_{-(n+1)}(k_t\rho)\right.$$

$$\left. + i^{-n+1}J_{-n+1}(k_t\rho)\right]\int_{-\infty}^{\infty}\frac{dk_\| q_\|(k_\|)e^{ik_\| z}}{(k_t^2 - iAk_\| + k_\|^2 B)} \quad .. \tag{29}$$

Note that the factor $e^{in\theta}$ has been suppressed in Eq. (29).

EVALUATION OF $q_\|(k_\|)$

Since we assumed that the beam is uniform along z, we obtain

$$q_\|(k_\|) = \frac{1}{2\pi}\int_{-L/2}^{L/2} e^{-ik_\| z}dz = \frac{1}{\pi k_\|}\sin\left(\frac{Lk_\|}{2}\right) \tag{30}$$

where L is the length of the pulse, and we measure distance with respect to the midplane of the pulse.

EVALUATION OF $q_t^{(n)}(k_t)$

MODEL I

$$q_t^{(0)}(k_t) = \int_0^R q^{(0)}\left(1 - \frac{\rho^2}{R^2}\right)J_0(k_t\rho)\ \rho\cdot d\rho$$

$$= q^{(0)}\frac{2}{k_t^2}\,J_2(k_t R) \tag{31}$$

$$q_t^{(1)}(k_t) = \int_0^R q^{(1)}\rho J_t(k_t\rho)\rho d\rho$$

$$= q^{(1)} \frac{R^2}{k_t} J_2(k_t R) \qquad (32)$$

MODEL II

$$q_t^{(0)}(k_t) = \int_0^R q^{(0)} \left(1 - \frac{\rho}{R}\right) J_0(k_t\rho)\rho d\rho$$

$$= q^{(0)} \left[-\frac{J_0(k_t R)}{k_t^2} + \frac{2}{Rk_t^3} \sum_{n=0}^{\infty} J_{2n+1}(k_t R)\right]^* \qquad (33)$$

$$q_t^{(1)}(k_t) = \int_0^R q^{(1)} J_1(k_t\rho)\rho d\rho$$

$$= q^{(1)} \left[-\frac{RJ_0(k_t R)}{k_t} + \frac{2}{k_t^2} \sum_{n=0}^{\infty} J_{2n+1}(k_t R)\right]^* \qquad (34)$$

EXPRESSIONS FOR $B_\theta^{(n)}(\rho, z)$

We will initially write down the appropriate expressions for $B_\theta^{(0)}$ and $B_\theta^{(1)}$ for each model separately and evaluate the same for the different cases as already mentioned.

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* In Eqs. (33) and (34), onecan use Struve functions instead of the infinite series as shown, when we recall that

$$\int_0^z J_0(z)dz = 2\sum_{n=0}^{\infty} J_{2n+1}(z) = zJ_0(z) + \frac{\pi z}{2}\left\{J_1(z)S_0(z) - J_0(z)S_1(z)\right\}$$

where $S_{0,1}$ stands for Struve function of order zero and 1, respectively.

MODEL I

Substitution of Eqs. (31) and (30) in (29) yields

$$B_\theta^{(0)}(\rho,z) = \frac{2\mu_0 c}{\pi}\ \frac{v}{c}\ q^{(0)}\int_0^\infty dk_t J_2(k_t R) J_1(k_t\rho)$$

$$\times \int_{-\infty}^{\infty} \frac{dk_\parallel}{k_\parallel}\ \frac{\text{Sin }(Lk_\parallel/2)\cdot e^{ik_\parallel z}}{(k_t^2 = iAk_\parallel + k_\parallel^2 B)}\ . \qquad (35)$$

In a similar fashion, after substituting Eqs. (32) and (30) in Eq. (29), one obtains

$$B_\theta^{(1)}(\rho,z) = \frac{\mu_0 c}{2\pi}\ \frac{v}{c}\ q^{(1)}\int_0^\infty dk_t\ \left[J_2(k_t\rho) - J_0(k_t\rho)\right] R^2 J_2(k_t R)\cdot k_t$$

$$\times \int_{-\infty}^{\infty} \frac{dk_\parallel}{k_\parallel}\ \frac{\sin\ (Lk_\parallel/2) e^{ik_\parallel z}}{(k_t^2 - iAk_\parallel + k_\parallel^2 B)}\ . \qquad (36)$$

We next proceed to evaluate the magnetic field amplitudes $B_\theta^{(0)}$ and $B_\theta^{(1)}$ for different velocity regimes of the beam chunk.

The integral over $k_\parallel$ in Eqs. (35) and (36) may be transformed, as was done earlier by Chandrasekhar, and is indicated below.

$$\mathcal{J}(z,k_t) = \int_{-\infty}^{\infty}\frac{dk_\parallel}{k_\parallel}\ \frac{\sin Lk_\parallel/2\ e^{ik_\parallel z}}{[(k_t^2 + k_\parallel^2 B) - iAk_\parallel]}$$

$$= \int_{-\infty}^{\infty}\frac{dk_\parallel}{k_\parallel}\ \frac{\sin Lk_\parallel/2\ e^{ik_\parallel z}}{(k_1^2 - iAk_\parallel)}$$

$$= \int_0^{\infty}\frac{dk_\parallel}{k_\parallel}\ \frac{1}{(k_1^4 + A^2 k_\parallel)}\Big[\{k_1^2 \sin\,[(z+L/2)k_\parallel] + k_\parallel A \cos\,[(z+L/2)k_\parallel]\}$$

$$- \{k_1^2 \sin\,[(z-L/2)k_\parallel] + Ak_\parallel \cos\,[(z-L/2)k_\parallel]\}\Big]\ . \qquad (37)$$

where

$$k_1^2 = k_t^2 + k_\parallel^2 B, \text{ and}$$

$$k_t^2 = k_x^2 + k_y^2$$

We will evaluate the field quantities in sequence.

CASE 1: $|v| = c$

In this approximation, $B = 0$, and

$$B_\theta^{(0)}(\rho, z) = \frac{2\mu_0 c}{\pi} q^{(0)} \int_0^\infty dk_t J_2(k_t R) J_1(k_t \rho) \int_{-\infty}^{\infty} \frac{dk_\parallel}{k_\parallel} \frac{\sin Lk_\parallel/2 \cdot e^{ik_\parallel z}}{(k_t^2 - iAk_\parallel)} \dots \tag{38}$$

Following Chandrasekhar, we note that the integral

$$\mathcal{J}(z,k_t) = \int_0^\infty \frac{dk_\parallel}{k_\parallel (k_t^4 + A^2 k_\parallel^2)} \Big[\{k_t^2 \sin[(z + L/2)k_\parallel] + Ak_\parallel \cos[(z + L/2)k_\parallel]\}$$

$$- \{k_t^2 \sin[(z - L/2)k_\parallel] + Ak_\parallel \cos[(z - L/2)k_\parallel]\} \Big] \dots \tag{39}$$

may be evaluated when one notes that

$$\int_0^\infty \frac{dx}{x} \frac{[\lambda^2 \sin \alpha x + Ax \cos \alpha x]}{(\lambda^4 + A^2 x^2)} = \begin{cases} \dfrac{\pi}{2\lambda^2}, & \text{for } \alpha \geq 0 \\[2ex] -\dfrac{\pi}{2\lambda^2} + \dfrac{\pi}{\lambda^2} e^{-|\alpha|\lambda^2/A}, & \text{for } \alpha < 0 \,. \end{cases} \tag{40}$$

From Eq. (40) we note that $\mathcal{J}(z;\, k_t) = 0$, for $z > L/2$, meaning that there is no electromagnetic disturbance ahead of the pulse, as it should be.

Employing the substitution $\xi_- = |z + L/2|$ and $\xi_+ = |z - L/2|$* we obtain

$$B_\theta^{(0)}(\rho, z) = 2\mu_0 c q^{(0)} \int_0^\infty \frac{dk_t}{k_t^2} J_2(k_t R) J_1(k_t \rho) \begin{cases} \left(1 - \exp - \dfrac{\xi_+ k_t^2}{A}\right), & \text{for } -L/2 < z < L/2 \\ \left[\exp - \dfrac{\xi_- k_t^2}{A} - \exp - \dfrac{\xi_+ k_t^2}{A}\right], & \text{for } z < -L/2 \end{cases} \quad (41)$$

We note that

$$\int_0^\infty dk_t J_2(k_t R) J_1(k_t \rho) k_t^{-2} = \begin{cases} \dfrac{R^2}{8\rho}, \ \ldots & \rho > R \\ \dfrac{\rho}{4}\left(1 - \dfrac{\rho^2}{2R^2}\right), \ \ldots & \rho \ \rho \ R \end{cases} \quad \ldots \quad (42)$$

It is also known that[3]

$$\int_0^\infty J_2(k_t R) J_1(k_t \rho) e^{-p^2 k_t^2} k_t^{-2} dk_t = \frac{R^3}{8p^2} \frac{\Gamma(1)}{\Gamma(3)\Gamma(2)} {}_3F_3\left(2, \frac{5}{2}, 1; 3\ 2, 4; \ -\frac{R^2}{p^2}\right) \ldots \quad (43)$$

If we use the notation $\zeta = \dfrac{R^2}{2p^2}$

$$\int_0^\infty J_2(k_t R) J_1(k_t \rho) e^{-p^2 k_t^2} k_t^{-2}\, dk_t = \zeta \frac{R}{8} \cdot {}_3F_3\left(2, \frac{5}{2}, 1; \ 3, 2, 4; \ -2\zeta\right) \ldots (44)$$

---

* This means that the field is measured along the coordinate ξ_- and ξ_+; that is, to the *back* and *front* ends of the pulse respectively.

3 G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Revised Edition, (1948).

Hence, we find that

$$B_\theta^{(0)}(R,z) = \frac{\mu_0 c q^{(0)} R}{4} \begin{cases} \left[1 - \zeta_+ \ {}_3F_3(2,\frac{5}{2},1;\ 3,2,4; -2\zeta_+)\right] & \text{for } -L/2 < z < L/2 \\ \left[\zeta_- \ {}_3F_3(2,\frac{5}{2},1;\ 3,2,4;\ -2\zeta_-) - \zeta_+ \ {}_3F_3(2,\frac{5}{2},1;\ 3,2,4;\ -2\zeta_+)\right] & \text{for } z < -L/2 \quad . \end{cases} \tag{45}$$

In Eqs. (43), (44), and (45), ${}_3F_3$ stands for a generalized hypergeometric function, which is defined as follows:

$${}_3F_3(\alpha_1,\alpha_2,\alpha_3;\ \beta_1,\beta_2,\beta_3;\ z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n}{n!(\beta_1)_n(\beta_2)_n(\beta_3)_n} z^n$$

and

$$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$$

$$(\alpha)_0 = 1$$

and

$$\xi_\pm = |z \mp L/2|$$

Note that in Eq. (45), $B_\theta^{(0)}$ has been evaluated at a radius equal to that of the beam, (that is, at $\rho = R$). We next note that, suppressing the factor $e^{i\theta}$,

$$B_\theta^{(1)}(\rho,z) = \frac{\mu_0 c}{2\pi} q^{(1)} R^2 \int_0^\infty dk_t k_t [J_2(k_t\rho) - J_0(k_t\rho)] J_2(k_t R)$$

$$\times \int_{-\infty}^{\infty} \frac{dk_\parallel}{k_\parallel} \frac{(\sin Lk_\parallel/2) e^{ik_\parallel z}}{(k_t^2 - iAk_\parallel)} \tag{46}$$

Employing the results of Eqs. (39) and (40), one finds that Eq. (46) may be written as

$$B_\theta^{(1)}(\rho, z) = \left(\frac{\mu_0 c}{2}\right) q^{(1)} R^2 \int_0^\infty \frac{dk_t}{k_t}[J_2(k_t\rho) - J_0(k_t\rho)] J_2(k_t R)$$

$$\times \begin{cases} [1 - \exp - \xi_+ k_t^2/A], \text{ for } - L/2 < z < L/2 \\ \\ [\exp - (\xi_- k_t^2/A) - \exp - (\xi_+ k_t^2/A)], \text{ for } z < - L/2 \end{cases} \quad (47)^*$$

Also

$$\int_0^\infty \frac{dk_t}{k_t} J_2(k_t R)[J_2(k_t\rho) - J_0(k_t\rho)] = \begin{cases} -\frac{1}{2} + \frac{3}{4}\frac{\rho^2}{R^2}, \text{ for } \rho < R \\ \\ \frac{1}{4}\frac{\rho^2}{R^2}, \text{ for } \rho > R \end{cases} \quad (48)$$

Since

$$\mathcal{J} = \int_0^\infty \frac{dk_t}{k_t} J_2(k_t R)[J_2(k_t\rho) - J_0(k_t\rho)] e^{-p^2 k_t^2} \quad (49)$$

does not appear to be easy to evaluate explicitly (except in the form of a series, probably), the integral in Eq. (49) will be evaluated for the case when $\rho = R$.

It has been shown earlier† that the integral (49) may be evaluated to be (for $\rho = R$)

$$-\mathcal{J}(\zeta) = \frac{1}{4} - \frac{1}{\zeta} Q_{\|}(\zeta) - 2Q_{22}(\zeta) \quad (50)$$

---

* Equation (47) has also been obtained earlier by Chandrasekhar.

† See reference 2.

where

$$Q_{\|}(\zeta) = \frac{1}{2}[1 - e^{-\zeta}\{I_0(\zeta) + I_1(\zeta)\}]$$

$$Q_{22}(\zeta) = \frac{1}{4}[1 - e^{-\zeta}\{I_0(\zeta) + 2I_1(\zeta) + I_2(\zeta)\}]$$

and,

$$\zeta_{\pm} = \frac{R^2 A}{2\xi_{\pm}}$$

Substitution of results given by relations (48) and (50) in Eq. (47) yields

$$-B_\theta^{(1)}(R,z) = \begin{cases} \mu_0 c q^{(1)} \dfrac{R^2}{2}\left[-\dfrac{1}{4} + \mathcal{S}(\zeta_+)\right], & \text{for } -L/2 < z < L/2 \\[2ex] \mu_0 c q^{(1)} \dfrac{R^2}{2}\left[[\mathcal{S}(\zeta_-) - \mathcal{S}(\zeta_+)]\right], & \text{for } z < -L/2 \end{cases} \qquad (51)^*$$

Equations (45) and (51) are the expressions for the field quantities we have been after. These expressions will be discussed later on.

CASE 2: $|v| \approx c$.

Again, we will assume here that the velocity $\mathbf{v}$ of the beam pulse is all z-directed. Recall now that† an integral of the following type is to be evaluated to obtain expressions for the field quantities $B_\theta^{(0)}$ and $B_\theta^{(1)}$.

$$\mathcal{S}(z;k_t) = \int_0^\infty \frac{dk_\|}{k_\|} \frac{[\{k_1^2 \sin[(z + L/2)k_\|] + Ak_\| \cos[(z + L/2)k_\|]\}]}{(k_1^4 + A^2k_\|^2)} \qquad (52)$$

where

$$k_1^2 = k_t^2 + k_\|^2 B, \text{ and } B = (1 - v^2/c^2) \quad .$$

---

* The sign here appears to be different from the one obtained by Chandrasekhar.

† See Eqs. (35) and (36).

Consider now the expression

$$(a^2 + x^3)\ (b^2 + x^2) \;=\; a^2b^2 + x^2(a^2 + b^2) + x^4 \qquad (53a)$$

and compare it with

$$k_t^4 + k_\parallel^4 B^2 + 2k_\parallel^2 k_t^2 B + A^2 k_\parallel^2 \;=\; B^2 \left\{ \frac{k_t^4}{B^2} + k_\parallel^4 + k_\parallel^2 \left(\frac{2k_t^2 B + A^2}{B^2} \right) \right\} \qquad (53b)$$

Comparing the coefficients in the expressions (53a) and (53b) and noting that $B \lll 1$, when $|v| \approx c$, one may conclude that (53b) may be approximated as

$$B^2 \left\{ \frac{k_t^4}{B^2} + k_\parallel^4 + k_\parallel^2 \frac{A^2}{B^2} \right\} \qquad , \text{ when } A^2 \gg 2k_t^2 B \;;$$

or, one may employ the following *approximate* substitution

$$a^2 \rightarrow \frac{A^2}{B^2} \qquad ; \qquad x \rightarrow k_\parallel$$

$$b^2 \rightarrow \frac{k_t^4}{A^2} \qquad ; \qquad a^2b^2 \rightarrow \frac{k_t^4}{B^2}$$

APPROXIMATION Ⓐ :

The approximation $A^2 \gg 2k_t^2 B$, implies either

(a) The conductivity of the plasma is large, or

(b) The transverse wave number is small (that is, the wavelength corresponding to the transverse wave number is large; that is, we are considering long wavelength disturbances, essentially).

One can show that the integral

$$\overset{Ⓐ}{\mathcal{J}}(\alpha;\lambda) = \int_0^\infty \frac{dx}{x} \frac{[(\lambda^2 + Bx^2)\sin \alpha x + Ax \cos \alpha x]}{(a^2 + x^2)(b^2 + x^2)}$$

$$= \frac{1}{B^2}\left[\frac{\lambda^2\pi}{2a^2b^2}\left\{1 + \frac{(b^2e^{-\alpha a} - a^2e^{-\alpha b})}{(a^2 - b^2)}\right\} + \frac{B\pi}{2}\left(\frac{e^{-\alpha b} - e^{-\alpha a}}{2(a^2 - b^2)}\right) + \frac{A\pi}{2(a^2 - b^2)}\left(\frac{e^{-\alpha b}}{b} - \frac{e^{-\alpha a}}{a}\right)\right]$$

for, $\alpha, a, b > 0$

(54)*

$$= \frac{1}{B^2}\left[\frac{-\lambda^2\pi}{2a^2b^2}\left\{1 + \frac{b^2e^{-|\alpha|a} - a^2e^{-|\alpha|b}}{(a^2 - b^2)}\right\} - \frac{B\pi}{2}\frac{e^{-|\alpha|b} - e^{-|\alpha|a}}{2(a^2 - b^2)} + \frac{A\pi}{2(a^2 - b^2)}\left(\frac{e^{-|\alpha|b}}{b} - \frac{e^{-|\alpha|a}}{a}\right)\right]$$

for $a, b > 0$ and $\alpha < 0$.

(55)*

That is, when $A^2 \gg 2k_t^2 B$, after some manipulation one finds that

$$\overset{Ⓐ}{\mathcal{J}}(\alpha;k_t) \approx \begin{cases} \dfrac{\pi}{2k_t^2}\left\{1 + \dfrac{k_t^4B^2}{A^4}e^{-\alpha A/B} + \dfrac{Bk_t^2}{2A^2}\left(e^{-\alpha k_t^2/A} - 3e^{-\alpha A/B}\right)\right\}, & \text{for } \alpha > 0 \\ -\dfrac{\pi}{2k_t^2} + \dfrac{\pi}{k_t^2}e^{-|\alpha|k_t^2/A} \\ -\dfrac{\pi}{k_t^2}\left\{\dfrac{k_t^4B^2}{A^4}e^{-|\alpha|A/B} + \dfrac{Bk_t^2}{2A^2}\left(e^{-|\alpha|k_t^2/A} + e^{-|\alpha|A/B}\right)\right\} & \text{for, } \alpha < 0 \end{cases}$$

(56)

---

* In the evaluation of (54) and (55) the following result due to Dr. C. Flammer was employed:

$$\int_0^\infty \frac{\sin mx\,dx}{x(a^2 + x^2)(x^2 + b^2)} = \frac{\pi}{2a^2b^2}\left\{1 + \frac{b^2e^{-ma} - a^2e^{-mb}}{(a^2 - b^2)}\right\}, \text{ for, } a, b, m > 0 .$$

However, we have to rewrite $\overset{(A)}{\mathcal{J}}(\alpha, k_t)$ as follows, because there should be no magnetic field ahead of the pulse.*

$$\overset{(A)}{\mathcal{J}}(\alpha, k_t) = \begin{cases} \dfrac{\pi}{2k_t^2}, & \text{for } \alpha > 0 \\ -\dfrac{\pi}{2k_t^2} + \dfrac{\pi}{k_t^2} e^{-|\alpha| k_t^2/A} - \dfrac{\pi B}{4A^2} e^{-|\alpha| k_t^2/A}, & \text{for, } \alpha < 0 . \end{cases} \tag{57}$$

The above result (57) is valid to first order in B.† We also recall that $\alpha = z \pm L/2$.

If we now use the notation

$$B_\theta^{(0)}(\rho, z)]_{|v| \approx c} = B_\theta^{(0)}(\rho, z)]_{|v| \equiv c} + B_\theta^{(0)'}(\rho, z) ,$$

and

$$B_\theta^{(1)}(\rho, z)]_{|v| \approx c} = B_\theta^{(1)}(\rho, z)]_{|v| \equiv c} + B_\theta^{(1)'}(\rho, z)$$

one finds that

$$\overset{(A)}{B_\theta^{(0)'}}(\rho, z) \approx \frac{2\mu_0 c \left(\frac{v}{c}\right) q^{(0)}}{\pi} \int_0^\infty J_2(k_t R) J_t(k_t \rho) dk_t \begin{cases} \dfrac{\pi B}{4A^2} \exp\left(-\dfrac{\xi_+ k_t^2}{A}\right), & -L/2 < z < L/2 \\ \dfrac{\pi B}{4A^2} \left[\exp\left(-\dfrac{\xi_+ k_t^2}{A}\right) - \exp\left(-\dfrac{\xi_- k_t^2}{A}\right)\right] & z < -L/2 \end{cases} \tag{58}$$

---

* The approximations employed here yield an erroneous result that there is a finite magnetic field ahead of the pulse, unless one uses the results indicated in (57).

† Note that $B \to 0$ as $v \to c$.

and

$$\underset{\text{Ⓐ}}{B_\theta^{(1)'}}(\rho, z) \approx \frac{\mu_0 c \frac{v}{c} q^{(1)} R^2}{2\pi} \int_0^\infty dk_t k_t [J_2(k_t\rho) - J_0(k_t\rho)] J_2(k_t R)$$

$$\times \begin{cases} \dfrac{\pi B}{4A^2} \exp\left(-\dfrac{\xi_+ k_t^2}{A}\right), & \text{for } -L/2 < z < L/2 \\[2ex] \dfrac{\pi B}{4A^2} \exp\left[\left(-\dfrac{\xi_+ k_t^2}{A}\right) - \exp\left(-\dfrac{\xi_- k_t^2}{A}\right)\right], & \text{for } z < -L/2 \quad . \end{cases} \tag{59}$$

Now, we note that $B_\theta^{(0)}$ and $B_\theta^{(1)}$ are zero for $z > L/2$ as it should be.

From Eqs. (58) and (59) we observe that the additional contributions to $B_\theta^{(0)}$ and $B_\theta^{(1)}$, when $|v| \approx c$ (under this approximation), are proportional to $(1 - v^2/c^2)$ and inversely proportional to A^2.* Since the magnitude of the additional term will be small as $v \to c$, an explicit calculation of $B_\theta^{(0)'}$ and $B_\theta^{(1)'}$ will not be carried out here.

$$^* \left.\begin{matrix} \dfrac{\underset{\text{Ⓐ}}{B_\theta^{(0)'}}}{B_\theta^{(0)}} \\[4ex] \dfrac{\underset{\text{Ⓐ}}{B_\theta^{(1)'}}}{B_\theta^{(1)}} \end{matrix}\right\} \sim \frac{B}{A^2} \; ; \; \left[B = \left(1 - \frac{v^2}{c^2}\right) \text{ and } A = \frac{1}{\epsilon_0} \frac{\sigma}{c} \frac{v}{c} \right] \; .$$

Approximation Ⓑ:

Let us now see what one obtains for $\mathcal{J}(z,k_t)$ in the approximation where $A^2 << 2k_t^2 B$; this approximation implies either

(a) The plasma is weakly ionized, or

(b) The transverse wave number is very large (that is, an extremely high frequency disturbance is understood).

In this approximation, one writes

$$(k_t^2 + k_{\|}^2 B)^2 + A^2 k_{\|}^2 \approx (k_t^2 + k_{\|}^2 B)^2 \quad . \qquad (60)$$

After a little algebra and simplification, one finds that

$$\overset{Ⓑ}{\mathcal{J}}(\alpha,\lambda) = \int_0^\infty \frac{dx[(\lambda^2 + Bx^2)\sin\alpha x + Ax\cos\alpha x]}{x(a^2 + x^2)^2}$$

$$= \begin{cases} \dfrac{\lambda^2\pi}{2a^4}\left(1 + \dfrac{2 + \alpha a}{2} e^{-\alpha a}\right) \\[2ex] + \dfrac{B\pi\alpha}{4a} e^{-\alpha a} + \dfrac{A\pi}{4a^3}(1 + \alpha a)e^{-\alpha a} \;, \quad \text{for } \alpha > 0 \\[2ex] -\dfrac{\lambda^2\pi}{2a^4}\left(1 + \dfrac{2 + |\alpha| a}{2} e^{-|\alpha| a}\right) \\[2ex] -\dfrac{B\pi|\alpha|}{4a} e^{-|\alpha| a} + \dfrac{A\pi}{4a^3}(1 + a|\alpha|)e^{-|\alpha| a} \;, \quad \text{for } \alpha < 0 \end{cases} \qquad (61)$$

where

$$a = \frac{\lambda}{\sqrt{B}} \quad .$$

Again, we rewrite $\overset{Ⓑ}{\mathcal{J}}(\alpha, k_t)$ as shown below to avoid the erroneous result that there is a finite magnetic field ahead of the pulse.* Noting that $B \to 0$, as $v \to c$, to first order in B, we obtain

* This appears to be inherent in the approximation.

$$\overset{\text{Ⓑ}}{\mathcal{J}}(\alpha, k_t) \approx \begin{cases} \dfrac{\pi}{2k_t^2} , & \text{for } \alpha > 0 \\ -\dfrac{\pi}{2k_t^2} - \dfrac{\pi}{2k_t^2}\left(1 + \dfrac{k_t\alpha}{\sqrt{B}}\right)\exp\left(-\dfrac{|\alpha| k_t}{\sqrt{B}}\right) , & \text{for } \alpha < 0 \end{cases} \tag{62}$$

Substituting the result (62) in Eqs. (35) and (36) we obtain for the *total* field amplitudes in this approximation the following expressions

$$\underset{\text{Ⓑ}}{B_\theta^{(0)}}(\rho, z) \approx 2\mu_0 c\left(\frac{v}{c}\right) q^{(0)} \int_0^\infty J_2(k_t R) J_1(k_t\rho) k_t^{-2} dk_t$$

$$\times \begin{cases} \left[1 + \dfrac{1}{2}\left(1 + \dfrac{k_t\xi_+}{\sqrt{B}}\right)\exp\left(-\dfrac{\xi_+ k_t}{\sqrt{B}}\right)\right] , & \text{for } -L/2 < z < L/2 \\ -\dfrac{1}{2}\left[\left(1 + \dfrac{k_t\xi_-}{\sqrt{B}}\right)\exp\left(-\dfrac{\xi_- k_t}{\sqrt{B}}\right) - \left(1 + \dfrac{k_t\xi_+}{\sqrt{B}}\right)\exp\left(-\dfrac{\xi_+ k_t}{\sqrt{B}}\right)\right] , & \text{for } z < -L/2 \end{cases} \tag{63}$$

and

$$\underset{\text{Ⓑ}}{B_\theta^{(1)}}(\rho, z) \approx \frac{R^2\mu_0 c\left(\frac{v}{c}\right) q^{(1)}}{2} \int_0^\infty dk_t [J_2(k_t\rho) - J_0(k_t\rho)] J_2(k_t R) k_t^{-1}$$

$$\times \begin{cases} \left[1 + \dfrac{1}{2}\left(1 + \dfrac{k_t\xi_+}{\sqrt{B}}\right)\exp\left(-\dfrac{\xi_+ k_t}{\sqrt{B}}\right)\right] & \text{for } -L/2 < z < L/2 \\ -\dfrac{1}{2}\left[\left(1 + \dfrac{k_t\xi_-}{\sqrt{B}}\right)\exp\left(-\dfrac{\xi_- k_t}{\sqrt{B}}\right) - \left(1 + \dfrac{k_t\xi_+}{\sqrt{B}}\right)\exp\left(-\dfrac{\xi_+ k_t}{\sqrt{B}}\right)\right] , & \text{for } z < -L/2 . \end{cases} \tag{64}$$

Because it appears that the integral

$$\int_0^{\infty} \frac{J_\mu(at)J_\nu(bt)e^{-\alpha t}dt}{t^\beta}$$

cannot be evaluated in closed form (μ, ν, α, and β are numbers) we will content ourselves with the following.

Since $B \to 0$ as $v \to c$, the exponent $\xi_\pm k/\sqrt{B}$ will be very large for any reasonable value of $\xi_\pm$. That is, for reasonable distances away from the pulse, the magnetic field amplitudes $B_\theta^{(0)}$ and $B_\theta^{(1)}$ will decay (rapidly) to zero. Clearly, this is what one would expect in a medium of negligible conductivity.

CASE 3: $|v| << c$

We again assume that the velocity $\mathbf{v}$ of the beam pulse is only z-directed. In considering this case, we note that [since $B = (1 - v^2/c^2)$] $B \approx 1$. Next let us assume that $A >> k_t^2$ which implies the plasma is highly conducting. In other words, one may use the approximation

$$k_t^4 + k_\parallel^2(2k_t^2 B + A^2) + k_\parallel^4 B^2 \approx k_t^4 + k_\parallel^4 + k_\parallel^2 A^2 \tag{65}$$

To evaluate the relevant field amplitudes, one needs to evaluate integrals of the type

$$\mathcal{J}(x,\lambda) = \int_0^{\infty} \frac{dx[\lambda^2 \sin \alpha x + x^2 \sin \alpha x + Ax \cos \alpha x]}{x(a^2 + x^2)(b^2 + x^2)}$$

that is, of the type already evaluated in (57). Hence, we obtain

$$\mathcal{J}(\alpha, k_t) \approx \begin{cases} \dfrac{\pi}{2k_t^2}, & \text{for } \alpha > 0 \\ \\ -\dfrac{\pi}{2k_t^2} + \dfrac{\pi}{k_t^2} e^{-|\alpha|(k_t^2/A)} - \dfrac{\pi}{4A^2} e^{-|\alpha|(k_t^2/A)}, & \text{for } \alpha < 0 \end{cases} \tag{66}$$

Employing (62), one can immediately write down the following expressions for the relevant field amplitudes

$$B_\theta^{(0)}(\rho,z) \approx \frac{2\mu_0 c\left(\frac{v}{c}\right)q^{(0)}}{\pi}\int_0^\infty J_2(k_t R)J_1(k_t\rho)dk_t$$

$$\times \begin{cases} \frac{\pi}{k_t^2}\left(1 - \exp - \frac{\xi_+ k_t^2}{A}\right) + \frac{\pi}{4A^2}\exp\left(-\frac{\xi_+ k_t^2}{A}\right), & \text{for } -L/2 < z < L/2 \\ \frac{\pi}{k_t^2}\left[\exp\left(-\frac{\xi_- k_t^2}{A}\right) - \exp\left(-\frac{\xi_+ k_t^2}{A}\right)\right] & \\ + \frac{\pi}{4A^2}\left[\exp\left(-\frac{\xi_+ k_t^2}{A}\right) - \exp\left(-\frac{\xi_- k_t^2}{A}\right)\right] & \text{for } z < -L/2 \end{cases} \tag{67}$$

and

$$B_\theta^{(1)}(\rho,z) \approx \frac{R^2\mu_0 c\left(\frac{v}{c}\right)q^{(1)}}{2\pi}\int_0^\infty dk_t[J_2(k_t\rho) - J_0(k_t\rho)]J_2(k_t R)$$

$$\frac{\pi}{k_t^2}\left[1 - \exp\left(-\frac{\xi_+ k_t^2}{A}\right) + \frac{\pi}{4A^2}\exp\left(-\frac{\xi_+ k_t^2}{A}\right)\right], \text{ for } -L/2 < z < L/2$$

$$\times \quad \frac{\pi}{k_t^2}\left[\exp\left(-\frac{\xi_- k_t^2}{A}\right) - \exp\left(-\frac{\xi_+ k_t^2}{A}\right)\right] \tag{68}$$

$$+ \frac{\pi}{4A^2}\left[\exp - \frac{\xi_+ k_t^2}{A} - \exp - \frac{\xi_- k_t^2}{A}\right], \text{ for } z < -L/2 \quad .$$

As was observed in Case 2, Approximation Ⓐ, we find the following from Eqs. (67) and (68). When $|v| << c$ and the plasma is highly conducting, except for the factor v/c (entering the expressions for the field quantities), there is an additional contribution to the field amplitudes which is quite small; this additional contribution varies as A^{-2}.

MODEL II

Recall now that in this model

$$q_t^{(0)} = \begin{cases} q^{(0)}(1 - \rho/R) \ , \ \text{for } \rho \le R \\ 0, \ \text{otherwise} \end{cases} \tag{13a}$$

and

$$q_t^{(1)} = \begin{cases} q^{(1)} e^{i\theta} \ , \ \text{for } \rho \le R \\ 0, \ \text{otherwise} \end{cases} \tag{13b}$$

We also have

$$q_\parallel(k_\parallel) = \frac{1}{\pi k_\parallel} \sin\left(\frac{Lk_\parallel}{2}\right) \ \ldots \tag{30}$$

$$q_t^{(0)}(k_t) = q^{(0)} \left[-\frac{J_0(k_t R)}{k_t^2} + \frac{2}{Rk_t^3} \sum_{n=0}^{\infty} J_{2n+1}(k_t R) \right] \ldots \tag{33}$$

and

$$q_t^{(1)}(k_t) = q^{(1)} \left[-\frac{RJ_0(k_t R)}{k_t} + \frac{2}{k_t^2} \sum_{n=0}^{\infty} J_{2n+1}(k_t R) \right] \ldots \tag{34}$$

Substitution of Eqs. (30) and (33) in (29) yields

$$B_\theta^{(0)}(\rho, z) = \frac{2\mu_0 c\left(\frac{v}{c}\right) q^{(0)}}{\pi} \int_0^\infty dk_t \left[-\frac{J_0(k_t R)}{k_t^2} + \frac{2}{Rk_t^3} \sum_{n=0}^{\infty} J_{2n+1}(k_t R) \right] J_1(k_t \rho) k_t^2$$

$$\times \int_{-\infty}^{\infty} \frac{dk_\parallel}{k_\parallel} \frac{\sin\left(\frac{Lk_\parallel}{2}\right) e^{ik_\parallel z}}{[k_t^2 - iAk_\parallel + k_\parallel^2 B]} \ \ldots \tag{69}$$

and

$$B_\theta^{(1)}(\rho,z) = \frac{\mu_0 c\left(\frac{v}{c}\right)q^{(1)}}{2\pi}\int_0^\infty dk_t[J_2(k_t\rho) - J_0(k_t\rho)]k_t^2\left[-\frac{RJ_0(k_tR)}{k_t} + \frac{2}{k_t^2}\sum_{n=0}^{\infty} J_{2n+1}(k_tR)\right]$$

$$\times\int_{-\infty}^{\infty}\frac{dk_\parallel}{k_\parallel}\frac{\sin\left(\frac{k_\parallel L}{2}\right)e^{ik_\parallel z}}{[k_t^2 - iAk_\parallel + k_\parallel^2 B]} \quad \ldots \qquad (70)$$

As mentioned in the Introduction, we will evaluate the expressions (69) and (70) for the case $|v| = c$ and $\rho = R$.

Since,

$$\left[-\frac{J_0(k_tR)}{k_t^2} + \frac{2}{Rk_t^3}\sum_{n=0}^{\infty} J_{2n+1}(k_tR)\right] = \frac{1}{k_t^2}J_2(k_tR) + \frac{2}{k_t^3R}\sum_{n=1}^{\infty} J_{2n+1}(k_tR) \quad \ldots \qquad (71)$$

we find that

$$B_\theta^{(0)}(\rho,z) = 2\mu_0 c q^{(0)}\int_0^\infty dk_t\left[\frac{J_2(k_tR)}{k_t^2} + \frac{2}{Rk_t^3}\sum_{n=1}^{\infty} J_{2n+1}(k_tR)\right]J_1(k_t\rho)$$

$$\times\begin{cases}\left[1 - \exp - \xi_+\frac{k_t^2}{A}\right], & \text{for } -L/2 < z < L/2 \\ \left[\exp\left(-\xi_-\frac{k_t^2}{A}\right) - \exp\left(-\xi_+\frac{k_t^2}{A}\right)\right], & \text{for } z < -L/2 \end{cases} \qquad (72)$$

Since the evaluation of the integrals is greatly simplified, we will consider the case only when $\rho = R$. One notes that the following results are needed to evaluate (72).

$$\int_0^\infty \frac{J_2(k_tR)J_1(k_tR)}{k_t^2}dk_t = \frac{R}{8} \quad \ldots \qquad (73a)$$

$$\int_0^\infty \frac{J_2(k_tR)J_1(k_tR)}{k_t^2}\exp(-p^2k_t^2)dk_t = \frac{R^3}{16p^2}\,{}_3F_3\left(2, \frac{5}{2}, 1; 3, 2, 4; -\frac{R^2}{p^2}\right) \qquad (73b)$$

$$\int_0^{\infty} \sum_{n=1}^{\infty} J_{2n+1}(k_t R) J_1(k_t R) k_t^{-3} dk_t = \frac{R^2}{48} \qquad (73c)$$

(using the result on p. 403 of Reference 3) and

$$\int_0^{\infty} \sum_{n=1}^{\infty} J_{2n+1}(k_t R) J_1(k_t R) k_t^{-3} \exp(-p^2 k_t^2) dk_t = \sum_{n=1}^{\infty} \left(\frac{R^2}{2p^2}\right)^n \frac{\Gamma(n)\left(\frac{R}{2}\right)^2}{2^n \Gamma(2n+2)}$$

$$\cdot \; {}_3F_3\left(\frac{2n+3}{2}, n+2, n; 2n+2, 2, 2n+3; -\frac{R^2}{p^2}\right) \ldots \qquad (73d)$$

where ${}_3F_3$ is the hypergeometric function defined as before. Substitution of the results (73a) to (73d) in Eq. (72) yields

$$B_\theta^{(0)}(R,z) = 2\mu_0 c q^{(0)} \begin{cases} \left[\frac{R}{6} - \frac{R}{8}\zeta_+ \; {}_3F_3\left(2, \frac{5}{2}, 1; 3, 2, 4; -2\zeta_+\right) - 2R \sum_{n=1}^{\infty} \frac{(\zeta_+)^n \Gamma(n)}{2^{n+2}\Gamma(2n+2)} \right. \\ \left. \cdot \; {}_3F_3\left(\frac{2n+3}{2}, n+2, n; 2n+2, 2, (2n+3); -2\zeta_+\right)\right] \\ \qquad -L/2 < z < L/2 \\ \frac{R}{8}\left\{\zeta_- \; {}_3F_3\left(2, \frac{5}{2}, 1; 3, 2, 4; -2\zeta_-\right) - \zeta_+ \; {}_3F_3\left(2, \frac{5}{2}, 1; 3, 2, 4; -2\zeta_+\right)\right\} \\ + R2 \left[\sum_{n=1}^{\infty} \frac{(\zeta_-)^n \Gamma(n)}{2^{n+2}\Gamma(2n+2)}\right. \\ \cdot \; {}_3F_3\left(\frac{2n+3}{2}, n+2, n; 2n+2, 2, 2n+3; -2\zeta_-\right) \\ - \sum_{n=1}^{\infty} \frac{(\zeta_+)^n \Gamma(n)}{2^{n+2}\Gamma(2n+2)} \\ \left. \cdot \; {}_3F_3\left(\frac{2n+3}{2}, n+2, n; 2n+2, 2, 2n+3; -2\zeta_+\right)\right] \\ \qquad z < -L/2 \end{cases}$$

(74)

where, as before

$$\zeta_{\pm} = \frac{R^2}{2p_{\pm}^2} = \frac{R^2 A}{2\xi_{\pm}}$$

$$\xi_{\pm} = \left| z \mp \frac{L}{2} \right|$$

and $R^2A/2$ is the relevant unit of distance.

One can write $B_\theta^{(1)}(\rho, z)$ as

$$B_\theta^{(1)}(\rho,z) = \frac{\mu_0 c q^{(1)}}{2} \int_0^\infty dk_t k_t^{-2}[J_2(k_t\rho) - J_0(k_t\rho)] \left[2 \sum_{n=0}^{\infty} J_{2n+1}(k_t R) - k_t R J_0(k_t R)\right]$$

$$\times \begin{cases} \left(1 - \exp - \dfrac{\xi_+ k_t^2}{A}\right), \text{ for } - L/2 < z < L/2 \\[2ex] \left[\exp\left(-\dfrac{\xi_- k_t^2}{A}\right) - \exp\left(-\dfrac{\xi_+ k_t^2}{A}\right)\right], \text{ for } z < - L/2 \end{cases} \qquad (75)$$

As in the evlauation of $B_\theta^{(0)}$[see Eq. (74)], we will assume $\rho = R$ and evaluate $B_\theta^{(1)}$, since this will simplify the algebra.

The following results are useful in the evaluation of (75), for $\rho = R$.

$$\int_0^\infty J_2(k_t R) \sum_{n=0}^{\infty} J_{2n+1}(k_t R) e^{-p^2 k_t^2} k_t^{-2} dk_t$$

$$= \frac{R}{4} \sum_{n=0}^{\infty} \frac{(\zeta)^{n+1}\Gamma(n+1)}{2^{n+2}\Gamma(2n+2)}$$

$$\cdot\ {}_3F_3\left(n+2, \frac{2n+5}{2}, n+1; 3, 2n+2, 2n+4; -2\zeta\right)$$

$$= Q^{(1)} \text{(say)} \qquad (76a)$$

$$\int_0^\infty J_2(k_t R) J_0(k_t R) k_t^{-1} e^{-p^2 k_t^2} dk_t = \frac{1}{8}\zeta\ {}_3F_3\left(\frac{3}{2}, 2, 1; 3, 1, 3; -2\zeta\right)$$

$$= Q^{(2)}, \text{ (say)} \qquad (76b)$$

$$\int_0^\infty J_0(k_t R)\sum_{n=1}^{\infty} J_{2n+1}(k_t R)e^{-p^2 k_t^2}\, k_t^{-2} dk_t$$

$$= \frac{R}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{\zeta}{2}\right)^n \Gamma(n)}{\Gamma(2n+2)}\, {}_3F_3\left(n+1, \frac{2n+3}{2}, n; 1, 2n+2, 2n+2; -2\zeta\right)$$

$$= Q^{(3)}, \text{ (say)} \tag{76c}$$

Note also that

$$-\int_0^\infty \frac{J_0(k_t\rho)2J_1(k_tR)dk_t}{k_t^2} + R\int_0^\infty \frac{dk_t}{k_t} J_0(k_t\rho)J_0(k_tR)$$

$$= -R\int_0^\infty \frac{J_0(k_t\rho)J_2(k_tR)dk_t}{k_t}$$

$$= 0, \text{ when } \rho = R$$

and

$$\int_0^\infty \frac{dk_t}{k_t^2}[J_0(k_tR) - J_0(k_tR)]\left[2\sum_{n=0}^{\infty} J_{2n+1}(k_tR) - k_tRJ_0(k_tR)\right] = \frac{R}{12}.$$

Hence, we obtain

$$B_\theta^{(1)}(R,z) = \frac{\mu_0 c q^{(1)}}{2}\begin{cases}\frac{R}{12} - 2[Q_+^{(1)} + Q_+^{(3)} - RQ_+^{(2)}], & \text{for } -L/2 < z < L/2\\[2ex] 2[Q_-^{(1)} + Q_-^{(3)} - RQ_-^{(2)} - Q_+^{(1)} - Q_+^{(3)} + RQ_+^{(2)}] & \text{for } z < -L/2\end{cases} \tag{77}$$

$Q_\pm$ involves the use of $\zeta_\pm$ in the above expressions.

As in the case of Model I, one can similarly obtain expressions for $B_\theta^{(0)}$ and $B_\theta^{(1)}$ for other cases where $|v| \neq c$.

However, we will not pursue this further; the interested reader can accomplish this after some algebra.

III DISCUSSION

EVALUATION OF FIELD COMPONENTS

One should evaluate numerically the field components from the formulae obtained in Eqs. (45) and (51) in the case of Model I, and Eqs. (74) and (77) in the case of Model II, and obtain appropriate corrections to the same.

Initially, we will obtain below the asymptotic values of Eqs. (45), (51), (74), and (77) and compare the same with Chandrasekhar's earlier results to find if the asymptotic values of the field components $B_\theta^{(0)}(R,z)$ and $B_\theta^{(1)}(R,z)$ are sensitive to beam charge density profile.

At large distance away from the pulse, ζ is a small quantity, and one can use the first term in the expansion of ${}_3F_3$. That is, Eq. (45) yields

$$\underset{\text{Lim } z\to-\infty}{B_\theta^{(0)}(R,z)_{(\mathrm{I})}} \to \frac{\mu_0 c q^{(0)} R}{4}(\zeta_- - \zeta_+)\ , \quad \text{to first order in } \zeta$$

$$\to \frac{\mu_0 c q^{(0)} R^3}{8}\left(\frac{\sigma}{c\epsilon_0}\right)\frac{L}{|z|^2} \quad (78)$$

Similarly, from Eq. (74)

$$\underset{\text{Lim } z\to-\infty}{B_\theta^{(0)}(R,z)_{(\mathrm{II})}} \to \frac{\mu_0 c q^{(0)} R}{4}\left[(\zeta_- - \zeta_+) + \frac{1}{3}(\zeta_- - \zeta_+)\right]$$

$$\to \frac{\mu_0 c q^{(0)} R^3}{8}\left(\frac{\sigma}{c\epsilon_0}\right)\left(\frac{4}{3}\right)\frac{L}{|z|^2} \quad (79)$$

In our units and notation, the asymptotic result obtained earlier by Chandrasekhar (where a different model was employed) is precisely the same as our Eq. (78). In a similar manner we find that from Eq. (51)

$$\lim_{z\to-\infty} B_\theta^{(1)}(R,z) \to \frac{\mu_0 c q^{(1)}}{32}\left(\frac{\sigma}{c\epsilon_0}\right) R^4 \, \frac{L}{|z|^2} \qquad (80)^*$$

and from Eq. (77)

$$\lim_{z\to-\infty} B_\theta^{(1)}(R,z) \to \frac{\mu_0 c q^{(1)}}{32}\left(\frac{\sigma}{c\epsilon_0}\right) \frac{1}{3} \cdot R^3 \cdot \frac{L}{|z|^2} \qquad (81)^*$$

Again, we note that the asymptotic value of $B_\theta^{(1)}(R,z)$ obtained by Chandrasekhar is precisely the same as our Eq. (80), since the *same* model was employed in both instances.

We conclude from the above Eqs. (78) to (81) that the asymptotic values of the field components *are sensitive to beam charge density profile*.

For the *same value of* $q^{(0)}$, since

$$\lim_{z\to-\infty} \underset{(II)}{B_\theta^{(0)}(R,z)} > \lim_{z\to\infty} \underset{(I)}{B_\theta^{(0)}(R,z)}$$

and

$$\lim_{z\to-\infty} \underset{(II)}{B_\theta^{(1)}(R,z)} < \lim_{z\to-\infty} \underset{(I)}{B_\theta^{(1)}(R,z)}$$

We note that for the same value of $q^{(0)}$ in the Models I and II of this report and that of Chandrasekhar, the value of beam current (total) in Model I is 4/3 of that in Model II, and the value of beam current in Chandrasekhar's model is twice that in Model II.

We will only remark that the growth rate of the hose instability will be lower with Model II beam than with Model I beam.

The evaluation of $B_\theta^{(0)}(R,z)$ and $B_\theta^{(1)}(R,z)$ as $z \to -L/2 - 0$ and $z \to L/2 - 0$, cannot be evaluated readily, since the function ${}_3F_3(\;)$, as $\zeta \to \infty$, cannot be approximated easily. Other representations will have to be examined.

---

* For the definition of appropriate values of $q^{(1)}$, see Eqs. (7b),(11b), and (13b). That is, for Model I, we have $|q^{(1)}| - [2d^{(1)}/R^2]\,q^{(0)}$, and for Model II, we have $|q^{(1)}| = [d^{(1)}/R]\,q^{(0)}$.

From the definition of $q^{(1)}$ it is clear that the decay distance of the magnetic field component $B_\theta^{(1)}$ is linearly related to the rigid displacement 'd'. For instance, in the case of Model I, if we take $d^{(1)}/R \approx 1/25$ (say), the decay distance of $B_\theta^{(1)}$ is approximately 1/50 of that of $B_\theta^{(0)}$ at large distances away from the pulse.

$$\frac{B_\theta^{(1)}}{B_\theta^{(0)}}_{Lim \to -\infty} \sim \frac{1}{50}$$

SOME NUMERICAL RESULTS

To develop a feeling for the decay distance, we will plot below approximately the behavior of $B_\theta^{(0)}(R,z)$ under the following conditions.

Assume: Beam radius = 0.5 cm (5×10^{-3} meters)
Speed of light $c = 3 \times 10^{10}$ cm/sec (3×10^{8} meters/sec)
Plasma conductivity $\sigma = 8.85 \times 10^{2}$ mhos (7.96×10^{12} e.s.u.)
Pulse is 30 meters long (corresponds to a pulse 0.1 μsec in time).

Then,

$$\zeta = \frac{R^2 A}{2\xi} \approx (4.166)\left(\frac{1}{\xi}\right) .$$

The behavior of $B_\theta^{(0)}(R,z)$ is plotted in Fig. 2.*

Note that the decay distance (that is, the distance in which the field decays to e^{-1} of its value) is of the order of 4 meters. This is the relevant unit of distance equal to $R^2A/2$.

COMMENTS ON COMPLEX CONDUCTIVITY, EXTERNAL MAGNETIC FIELDS, ETC.

The following comments are in order.

The treatment employed here assumes that the conductivity σ of the plasma is a purely real quantity.

* Equations (45) and (51) have been employed for this plot in Fig. 2, $B_\theta^{(0)}(R,z)]_n = \{[\mu_0 c q^{(0)}(R/4)]^{-1} B_\theta^{(0)}(R,z)\}$ and the normalized value of $B_\theta^{(1)}(R,z)$ are plotted.

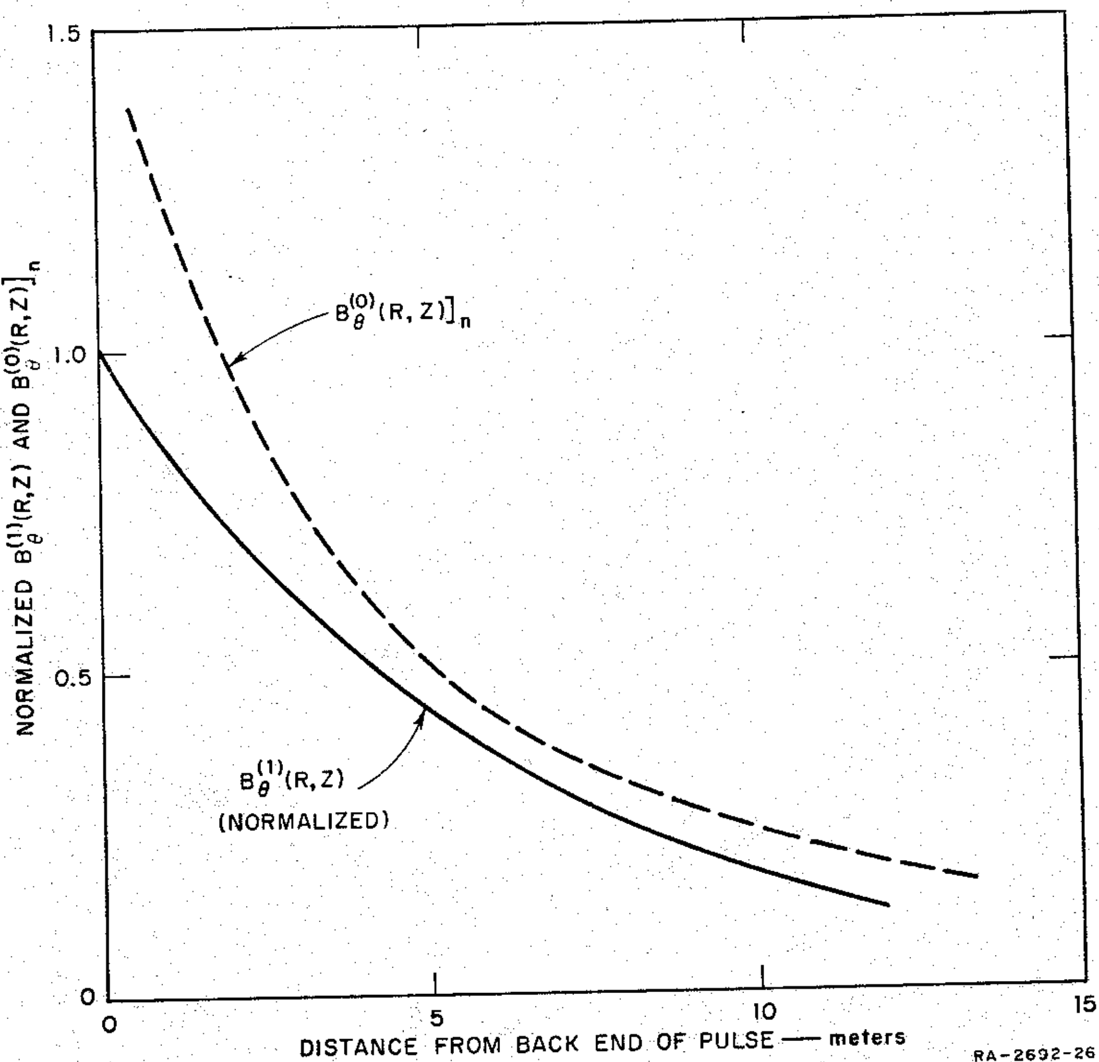

FIG. 2 BEHAVIOR OF $B_\theta^{(0)}(R,z)$ AND $B_\theta^{(1)}(R,z)$ FOR THE EXAMPLE CONSIDERED (See Text)

However, in practice one observes that the conductivity σ is a complex quantity. To treat such a case one has to evaluate an integral of the type shown in Eq. (37) which appears to be extremely complicated; also, for any value of beam pulse velocity v such that $0 < |v/c| < 1$, one has to evaluate Eq. (37) before the final integration over k_t can be carried out.

Note that the tensor character of the conductivity σ (because of the presence of the magnetic field due to the beam pulse) should be considered in a more complete treatment of the problem.

IV CONCLUDING REMARKS

The sensitivity of decay distances of magnetic field components $B_\theta^{(0)}(R,z)$ and $B_\theta^{(1)}(R,z)$ (due to a finite beam pulse penetrating a conducting plasma) to beam charge density profile has been investigated. Suitable corrections have been obtained in connection with one of the Models for the cases when the pulse velocity v satisfies the restrictions $|v/c| \approx 1$ and $|v/c| \ll 1$, under some conditions. Also, some numerical results have been plotted in Fig. 2 to show the general behavior of the magnetic field components $B_\theta^{(0)}$ and $B_\theta^{(1)}$ in connection with one of the models.

Finally, as mentioned before in Section III the complex and/or tensor character of the plasma conductivity σ for values of v/c such that $0 < |v/c| < 1$ should be treated in future work.

ACKNOWLEDGMENT

The writer would like to thank Dr. C. Flammer for some critical comments.